Solving a Dirichlet problem for Poisson's Equation on a disc is as hard as integration.

Akitoshi Kawamura, Florian Steinberg, Martin Ziegler

Technische Universität Darmstadt

August 1, 2013

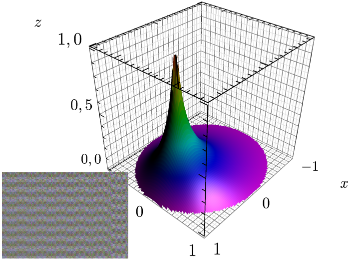


Table of contents

Definitions and examples

Recall that dyadic number is a number of the form $\frac{r}{2^n}$ for some $r \in \mathbb{Z}$ and $n \in \mathbb{N}$.

Definition

A real number x is called computable if there is a computable sequence $(d_n)_{n\in\mathbb{N}}$ of dyadic numbers, such that $|x - d_n| \leq 2^{-n}$ for every n. It is called polytime computable if there is such a sequence which computable in time polynomial in the value of n.

Examples

- π, e and $\ln(2)$ are polytime computable.
- It is not hard to construct uncomputable reals, computable reals not computable in polytime, etc.

Let $f : [0,1] \to \mathbb{R}$ be a function. A function $\mu : \mathbb{N} \to \mathbb{N}$ satisfying

$$|x - y| \leq 2^{-\mu(n)} \quad \Rightarrow \quad |f(x) - f(y)| \leq 2^{-n}$$

for all $x, y \in [0,1]$ is called *modulus of continuity* of f.

Example

1. Any Hölder continuous function has a linear modulus of continuity.
2. The function

$$f : x \mapsto \begin{cases} \frac{1}{1-\ln(x)} & , \text{ if } x \neq 0 \\ 0 & , \text{ if } x = 0 \end{cases}$$

does not have a polynomial modulus of continuity.

Definition

A real function $f : [0,1] \to \mathbb{R}$ is called computable, iff

1. f has a computable modulus of continuity.
2. the sequence of values of f on dyadic arguments is computable.

It is called polytime computable if

1. it has a polynomial modulus of continuity.
2. there is a machine which, upon input $\langle d, 1^n \rangle$, returns a dyadic number s such that $|f(d) - s| \leq 2^{-n}$ in polynomial time.

Example

A constant function is (polytime) computable iff its value is.

Corollary (Main Theorem of computable Analysis)

Any computable function is continuous.

Example

The function

$$f : [0,1] \to \mathbb{R}, x \mapsto \begin{cases} -x\ln(x) & \text{,if } x \neq 0 \\ 0 & \text{,if } x = 0 \end{cases}$$

is polytime computable.

Proof.

One can check, that $n \mapsto 2(n+1)$ is a modulus of continuity. The function ln is computable on the interval $[2^{-N}, 1]$ in time polynomial in the precision and N. For dyadic input we can now make the case distinction $d = 0$ or $d \geq 2^{-N}$ and compute the function. $\square$

Complexity of integration

Recall that $\mathcal{NP}$ is the class of polynomial time verifiable problems.
Prototype:

$$B = \left\{ x \mid \exists y \in \{0,1\}^{p(|x|)} : \langle y, x \rangle \in A \right\}.$$

Example

Many problems are known to be $\mathcal{NP}$ complete, for example SAT.

The question whether $\mathcal{P} = \mathcal{NP}$ is wide open and considered one of the big questions of modern mathematics.
For a fixed Element $x \in B$, there may be multiple witnesses, that is $y \in \{0,1\}^{p(|x|)}$ such that $\langle y, x \rangle \in A$.

Definition

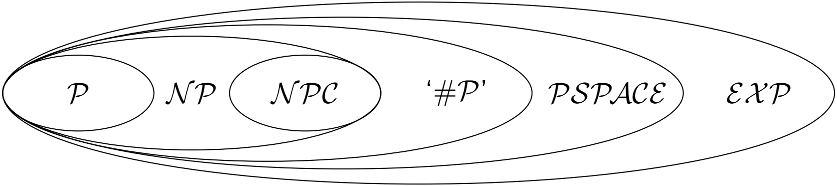
A function $\psi : \mathbb{N} \to \mathbb{N}$ is called $\#\mathcal{P}$ computable, if there is a polynomial time computable set A and a polynomial p such that

$$\psi(x) = \#\{y \in \{0,1\}^{p(|x|)} \mid \langle y, x \rangle \in A\}.$$

The following are easy to see:

Lemma

1. $\mathcal{FP} \subseteq \#\mathcal{P}$.
2. $\mathcal{FP} = \#\mathcal{P}$ *implies* $\mathcal{P} = \mathcal{NP}$.



Theorem (Friedman (1984), Ko (1991))

The following are equivalent:

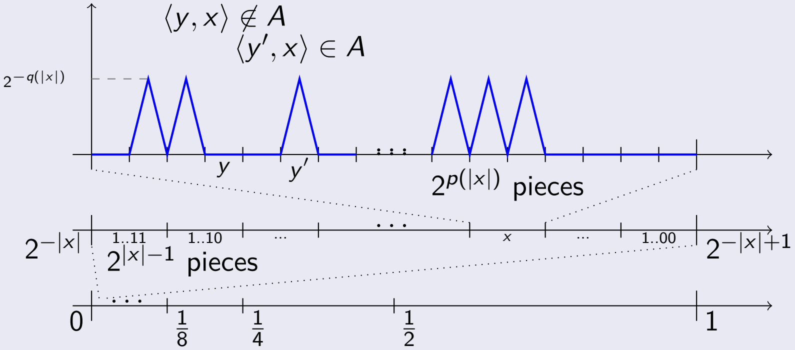
1. *The indefinite integral over each polytime computable function is a polytime computable function.*
2. $\mathcal{FP} = \#\mathcal{P}$
3. *The indefinite integral over each smooth, polytime computable function is a polytime computable function.*

proof sketch $1 \Leftrightarrow 2$.

'$\Leftarrow$': Standard grid approach: It is possible to verify in polynomial time, that a square lies beneath the function. Now $\mathcal{FP} = \#\mathcal{P}$ implies, that we can already count these squares in polynomial time. With help of the modulus of continuity an approximation to the integral can be given. $\square$

proof sketch 1 $\Leftrightarrow$ 2.

'$\Rightarrow$': Let $\psi(x) = \#\{y \in \{0,1\}^{p(|x|)} \mid \langle y, x\rangle \in A\}$. Consider the following polytime computable function h_ψ:



$\psi(x)$ can be read from the binary expansion of the integral over an appropriate interval in polynomial time. The polynomial q can be chosen such that h_ψ and even $\frac{h_\psi}{x}$ are Lipschitz continuous. $\square$

Corollary

The following are equivalent:

1. *For any polytime computable* $f : [0,1] \times [0,1] \to \mathbb{R}$ *the function*
$$x \mapsto \int_{[0,1]} f(x,y)dy$$
is again polytime computable.
2. $\mathcal{FP} = \#\mathcal{P}$.

proof (sketch).

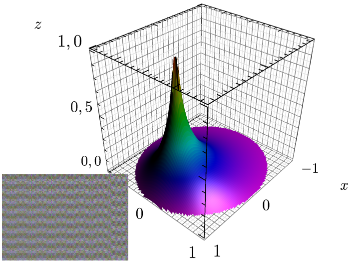
exactly the same ideas as in the previous proof:

2. $\Rightarrow$ 1. Using a similar grid approach.

1. $\Rightarrow$ 2. Again by specifying a suitable function.

$\square$

Solving a Dirichlet problem for Poisson's Equation on a disc is as hard as integration.



Consider the partial differential equation

$$\Delta u = f \text{ in } B_d, \quad u|_{\partial B_d} = 0.$$

We want to sketch a proof of the following:

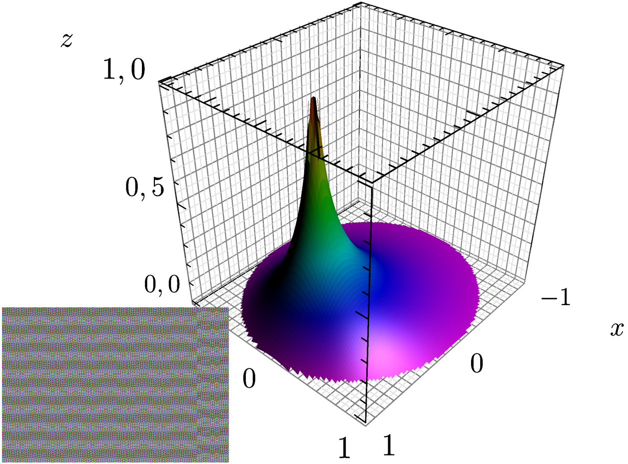
Theorem (Kawamura, S., Ziegler, 2013)

The following statements are equivalent:

1. $\mathcal{FP} = \#\mathcal{P}$
2. *The unique solution u is polytime computable whenever f is.*

For the proof we will restrict our attention to the case $d = 2$. Furthermore, we will identify $\mathbb{R}^2$ with $\mathbb{C}$ and heavily use the classical solution formula in terms of the Green's function.

$$u(z) = \int_{B_2} \underbrace{-\frac{1}{2\pi}\left(\ln\left(|w-z|\right) - \ln\left(|wz^* - 1|\right)\right)}_{=:G(w,z)} f(w)dw$$



Proof (of the Theorem) '$\Rightarrow$'.

It is not hard to see, that u has a linear modulus of continuity whenever f is bounded.
Let d be a (complex) dyadic number. If $|d|$ is too close to 1, return zero. If not, set $\delta \approx (1 - |d|)/2$, $B := B_2(d, \delta)$ and return approximations to

$$\int_{B_2 \setminus B} \ln(|w - d|) f(w) dw$$
$$- \int_{B_2} \ln(|wd^* - 1|) f(w) dw$$
$$+ \int_0^\delta r \ln(r) \int_0^{2\pi} f(re^{i\varphi} + d) d\varphi dr$$

(scaled by $-\frac{1}{2\pi}$), which is possible in polynomial time. $\square$

Proof (of the Theorem) '$\Leftarrow$'.

From the proof of the theorem about the complexity of integration, one can see that it suffices to integrate the 'bump functions' h_ψ. For such a function set

$$f(w) := \frac{h_\psi(|w|)}{|w|}.$$

Since f and Δ are radially symmetric, also u will be radially symmetric. Transforming Poisson's equation to polar coordinates now results in

$$(ru')' = rf = h$$

Therefore, the integral of h_ψ can be recovered from u'. For the derivative to be polytime computable we need a bound for the second derivative. This can be extracted by tedious computations from the solution formula, whenever f is Hölder continuous. $\square$

Thank you!